1st Grade
1st Semester
Calculus 1
Mathematics
Ms RAHMOUN

The Complex Numbers

1/ Vocabulary

عدد عقدي	Complex number	nombre complexe
رأس - رؤوس (مضلع)	Vertex - vertices	un sommet, des sommets
حقل (جبري)	a field	un corps algébrique
مزوّد بـ ...	Provided with ...	muni de ...
عملية داخلية	intern operation	opération interne
الجزء الحقيقي	real part	la partie réelle
الجزء التخيلي الصرف	purely imaginary part	partie purement imaginaire
المرافق	The conjugate	le conjugué
الطويلة و العمدة	Modulus and argument	module et argument
الصيغة الجبرية	Algebric form	forme algébrique
الصيغة المثلثية	trigonemetric form	forme trigonométrique
الصيغة الأسية	exponential form	forme exponentielle
الجذور التربيعية	Square roots	les racines carrées
الجذر النوني	nth root	racine énième

2/ The Complex number field $\mathbb{C}$

We call field of Complex numbers and we denote by $\mathbb{C}$, the set $\mathbb{R}^2$ provided with the two following intern operations:

$$\forall (x,y),(x',y') \in \mathbb{R}^2 \quad (x,y)+(x',y') = (x+x', y+y')$$

$$(x,y)\cdot(x',y') = (xx'-yy', xy'+x'y)$$

1°/ We identify any real $x \in \mathbb{R}$ with its complex writing $(x,0)$. We then write $x \in \mathbb{C}$ meaning $(x,0) \in \mathbb{C}$

2°/ We denote the complex $(0,1)$ by i, so $i=(0,1)$

3°/ Since, $\forall (x,y) \in \mathbb{R}$: $(x,y) = (x,0) + (0,1)(y,0)$ according to the new operations $+$ and $\cdot$, then we may write:

$$(x,y) = x + iy = z$$

4°/ $i^2 = (0,1)\times(0,1) = \underbrace{(-1,0) = -1}_{\text{By abuse of writing.}}$

a) The Algebric (or Cartesian) form

$$\forall Z \in \mathbb{C},\ \exists!\ (x,y) \in \mathbb{R} /\ Z = x + iy$$

1 °/ x is called the real part of Z, and is noted $Re(Z)$

2 °/ y is called the imaginary part of Z, and is noted $Im(Z)$

3 °/ $Z \in \mathbb{C}$ is called real iff $Im(Z) = 0$

4 °/ $Z \in \mathbb{C}$ is called purely imaginary iff $Re(Z) = 0$

5 °/ $\forall Z, Z' \in \mathbb{C}$ $\quad Z = Z' \iff \begin{cases} x = x' \\ y = y' \end{cases}$

6 °/ $\forall Z \in \mathbb{C}$, $\quad Z^2 = -1 \iff Z \in \{i, -i\}$

7 °/ $\begin{cases} i^0 = 1 \\ i^1 = i \\ i^2 = -1 \\ i^3 = -i \end{cases}$ and in general $\forall n \in \mathbb{N},\ i^{n+4} = i^n$

⊛ The Conjugate:

Let $Z = x + iy \in \mathbb{C}$. The Conjugate of Z is the Complex number $\overline{Z}$ defined by $\overline{Z} = x - iy$

⊛ Properties of the Conjugate:

1 °/ $\forall Z \in \mathbb{C}$, $\overline{\overline{Z}} = Z$; $Re(Z) = \frac{Z + \overline{Z}}{2}$; $Im(Z) = \frac{Z - \overline{Z}}{2}$.

2 °/ $\forall Z_1, Z_2 \in \mathbb{C}$ $\quad \overline{Z_1 + Z_2} = \overline{Z_1} + \overline{Z_2}$ and $\begin{cases} \overline{Z_1 \cdot Z_2} = \overline{Z_1} \cdot \overline{Z_2} \\ \overline{\left(\frac{Z_1}{Z_2}\right)} = \frac{\overline{Z_1}}{\overline{Z_2}} ; (Z_2 \neq 0) \end{cases}$

3 °/ $\forall Z \in \mathbb{C}$ $\quad Z = \overline{Z} \iff Z \in \mathbb{R}$

and $Z = -\overline{Z} \iff Z$ is purely imaginary.

Example: write in the algebric forme, the complex: $Z = (3 + 6i)^2$. then find it conjugate.

Well, $Z = (3 + 6i)^2 = (3)^2 + (6i)^2 + 2(3)(6i)$

$= 9 - 36 + 36i = -27 + 36i$

$\overline{Z} = -27 - 36i$

Remark: Don't mix up the Conjugate $\overline{Z} = x - iy$ with the Opposit of Z which is $-Z = -x - iy$ or with the reverse of Z which is $\frac{1}{Z} = \frac{\overline{Z}}{|Z|^2} = \frac{x}{x^2+y^2} - i\frac{y}{x^2+y^2}$ (here, $Z \neq 0$ of course)

b) The trigonometric form:

• Modulus and Argument:

Modulus → Let $z = x+iy \in \mathbb{C}$. The modulus of z is the Quantity:

$|z| = \sqrt{x^2+y^2}\ (\geq 0)$. It verifies the following, $\forall z \in \mathbb{C}$:

1 °/ $|\bar{z}| = |z|$ and $z\bar{z} = |z|^2$ (or $|z| = \sqrt{z\bar{z}}$).

2 °/ If $z \neq 0$ then $\dfrac{1}{z} = \dfrac{\bar{z}}{|z|^2}$

3 °/ $|z| = 0 \iff z = 0$

And $\forall z, z' \in \mathbb{C}$

1 °/ $|zz'| = |z| \cdot |z'|$

2 °/ $\left|\dfrac{1}{z}\right| = \dfrac{1}{|z|}$; $\left|\dfrac{z}{z'}\right| = \dfrac{|z|}{|z'|}$.

3 °/ $|z+z'| \leq |z| + |z'|$ and in general:

$$\left|\sum_{1}^{n} z_k\right| \leq \sum_{1}^{n} |z_k|$$

Remark: When z is real, it modulus is exactly it absolute value. that is why they have the same notation $|\cdot|$.

Important property of the modulus:

$$\boxed{\exists \lambda \in \mathbb{R}^{+} \,/\, z' = \lambda z \iff |z| + |z'| = |z+z'|}$$

Argument → Let $z = x+iy \in \mathbb{C}^{*}$, (i.e. $z \neq 0$).

We call argument of z and note $\arg(z)$ the angle θ that verifies:

$$\begin{cases} \cos\theta = \dfrac{x}{|z|} \\ \sin\theta = \dfrac{y}{|z|} \end{cases}$$

Remarks

1 °/ $\mathrm{Arg}(0)$ doesn't exist.

2 °/ if $\arg(z) = \theta$ then $\theta + 2k\pi$, $(k \in \mathbb{Z})$ is also an argument of z. We write $\mathrm{Arg}\, z \equiv \theta\ [2\pi]$.

important 3 °/ $\forall z \in \mathbb{C}^{*}$, $\exists! \alpha \in [0, 2\pi]$ / $\mathrm{Arg}(z) = \alpha$

The trigonometric form

Let be $z \in \mathbb{C}^*$, $z = x+iy$.

The trigonometric form of z is given by:

$$\boxed{z = |z| \left(\cos\theta + i\sin\theta\right)}$$

where $\arg(z) \equiv \theta \quad [2\pi]$.

⚠ One has: $\forall z_1, z_2 \in \mathbb{C}^*$,

1./ $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$

2./ $\arg\left(\frac{1}{z_1}\right) = -\arg(z_1) = \arg(\bar{z}_1)$

3./ $\arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg z_2$

4./ $\arg((z_1)^n) = n . \arg(z_1)$ i.e. The Moivre formula: $\boxed{(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta)}$ important

5./ $\boxed{z_1 = z_2 \iff \begin{cases} |z_1| = |z_2| \\ \arg z_1 \equiv \arg z_2 \quad [2\pi] \end{cases}}$

Exercises

At home

1. Find the algebric form of the Complex numbers:

$$z_1 = \frac{3+6i}{3-4i} \quad ; \quad z_2 = \frac{2+5i}{1-i} + \frac{2-5i}{1+i}$$

2. write in the trigonometric form then in the algebric form the Complex of modulus 3 and argument $\left(-\frac{\pi}{3}\right)$

3. Find the modulus and the argument of:

$$e^{e^{i\alpha}} \quad \text{and} \left(e^{i\theta} + e^{2i\theta}\right).$$

After the exponential form

At the amphi

1. Find the algebric form of the Complex number:

$$z = \left(\frac{1+i}{2-i}\right)^2$$

2. Find the modulus and the argument of the following Complex numbers:

* $U = \frac{\sqrt{6} - i\sqrt{2}}{2}$

* $V = 1 - i$

* $W = U^8$

3. Using the Moivre formula find $\cos(2x)$ and $\sin(2x)$ in of $\cos x$ and $\sin x$.

C) The exponential form

$\forall \theta \in \mathbb{R}$, One has $e^{i\theta} = \cos\theta + i\sin\theta$ ✓ Very important. Called The complex exponential.

So, when $z \in \mathbb{C}^*$, $z = |z|(\cos\theta + i\sin\theta)$, then

$$\boxed{z = |z|\, e^{i\theta}}$$

This is called the exponential form (or the Euler form) of the complex z.

Usual examples

Algebric form	exponential form
1./ 1	$e^{i(0)}$
2./ i	$e^{i(\frac{\pi}{2})}$
3./ $-i$	$e^{i(\frac{3\pi}{2})}$
4./ -1	$e^{i(\pi)}$
5./ $\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}$	$e^{i(\frac{\pi}{4})}$
6./ $\frac{1}{2} + i\frac{\sqrt{3}}{2}$	$e^{i(\frac{\pi}{3})}$
7./ $\frac{\sqrt{3}}{2} + i\frac{1}{2}$	$e^{i(\frac{\pi}{6})}$

⚠ **Remark**

$\forall \theta, \theta' \in \mathbb{R}$:

1) $e^{i\theta} \cdot e^{i\theta'} = e^{i(\theta+\theta')}$

2) $e^{-i\theta} = \frac{1}{e^{i\theta}} = \overline{e^{i\theta}}$

3) $\left(e^{i\theta} = e^{i\theta'}\right) \Leftrightarrow \left(\exists k \in \mathbb{Z} / \theta = \theta' + 2k\pi\right)$

⊛ The Euler formula:

$$\forall \theta \in \mathbb{R}, \quad \cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} \; ; \quad \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

Proof: very simple! Let $\theta \in \mathbb{R}$,

$$e^{i\theta} = \cos\theta + i\sin\theta \quad \ldots (1)$$

$$e^{-i\theta} = \cos(-\theta) + i\sin(-\theta)$$

$$= \cos\theta - i\sin\theta. \quad \ldots (2)$$

- adding (1) and (2) gives: $e^{i\theta} + e^{-i\theta} = 2\cos\theta$

$\Rightarrow \cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$

- Subtracting (1) and (2) gives the Sin formula.

At home: Use Euler formula to find $\cos^2\theta$ and $\sin^2\theta$, $\forall \theta \in \mathbb{R}$.

3/ Solving Equations in $\mathbb{C}$

a) First order equations:

Let $z, a, b \in \mathbb{C}$ with $a \neq 0$

$$\boxed{az + b = 0 \iff z = -\frac{b}{a}}$$

b) Second order equations

Let $z \in \mathbb{C}$, $a, b, c \in \mathbb{R}$ / $a \neq 0$.

The equation $az^2 + bz + c = 0$ admits exacty two roots (solutions) z_1, z_2 such that:
$$\begin{cases} z_1 + z_2 = -\frac{b}{a} \\ z_1 \cdot z_2 = \frac{c}{a} \\ a(z - z_1)(z - z_2) = 0 \end{cases}$$

To find z_1 and z_2 easiely, it is common to use the discriminant $\Delta = b^2 - 4ac$.

Δ admits exactly two square roots, say δ and $(-\delta)$.

Then,
$$\boxed{z_1 = \frac{-b-\delta}{2a}, \quad z_2 = \frac{-b+\delta}{2a}}$$

⚠ $\boxed{\Delta = \delta^2}$

It remains to learn:

Important → **How to find the square roots of a complex number?**

Let $z = a + ib$ we want to find w / $w^2 = z$

Put $w = x + iy$ then $w^2 = (x+iy)^2 = x^2 - y^2 + i(2xy)$

$w^2 = z \iff x^2 - y^2 + 2xyi = a + ib$

→ By analogy, we get $\begin{cases} x^2 - y^2 = a \\ 2xy = b \end{cases}$

→ But, we may add another condition which is:

$w^2 = z \iff |w|^2 = |z|$, this means that:

$$x^2 + y^2 = \sqrt{a^2 + b^2}$$

Finely, we solve the system: $\begin{cases} x^2 - y^2 = a \\ x^2 + y^2 = \sqrt{a^2+b^2} \\ (xy) \text{ has the same sign as } b \end{cases}$

Examples a/ Find square roots of $z = 8 - 6i$.

Let $w = x + iy$ / $w^2 = z$

$$(x+iy)^2 = 8 - 6i \iff \begin{cases} x^2 - y^2 = 8 \\ 2xy = -6 \end{cases}$$

$$|w|^2 = |z| \iff x^2 + y^2 = 10.$$

So $\begin{cases} x^2 - y^2 = 8 \\ x^2 + y^2 = 10 \\ 2yx = -6 \end{cases} \iff \begin{cases} 2x^2 = 18 \\ 2y^2 = 2 \\ x.y < 0 \end{cases} \Rightarrow \begin{cases} x = \pm 3 \\ y = \pm 1 \\ x.y < 0 \end{cases}$

done $w_1 = 3 - i$ and $w_2 = -3 + i$ are square roots of z

b/ Solve in $\mathbb{C}$ the second order equation,

$$2z^2 + 4\sqrt{2}\, z + 3i = 0$$

One has $\Delta = (4\sqrt{2})^2 - 4(2).(3i)$

$= 2.16 - 4.6.i = 4.(8 - 6i)$

So $\sqrt{\Delta} = 2(3 - i)$ or $2(-3 + i)$

Yields $z_1 = \dfrac{-4\sqrt{2} - 2(3-i)}{2(2)} = -\dfrac{(2\sqrt{2}+3)}{2} + \dfrac{1}{2}i$

and

$z_2 = \dfrac{-4\sqrt{2} + 2(3-i)}{2(2)} = -\dfrac{(2\sqrt{2}-3)}{2} - \dfrac{1}{2}i$

c) **Nth order equations (nth root of Z)**

Let $(Z$ and $L) \in \mathbb{C}^*$, $n \in \mathbb{N}^*$. ⚠ $Z^n = 0 \iff Z = 0$

To solve the equation $Z^n = L$, the exponential writing is more suitable.

1/ We begin by writing L in it exponential form,

$L = |L| e^{i\theta}$.

2/ Then identify: $z = |z| e^{i\alpha} \Rightarrow z^n = |z|^n e^{in\alpha}$ with L

3/ We get $\begin{cases} |z|^n = |L| \\ n\alpha \equiv \theta \quad [2\pi] \end{cases}$

So $\begin{cases} |z| = (|L|)^{\frac{1}{n}} \\ \alpha = \frac{\theta}{n} + \frac{2k\pi}{n} \quad k \in \mathbb{Z} \end{cases}$

⚠ But beware, we have exactly n nth roots not an infinity. Let us proove it.

Not more than n: Put $z_k = |L|^{\frac{1}{n}} e^{i\left(\frac{\theta}{n} + \frac{2k\pi}{n}\right)} \quad k \in \mathbb{Z}$

and Calculate $z_{k+n} = |L|^{\frac{1}{n}} \cdot e^{i\left[\frac{\theta}{n} + \frac{2(k+n)\pi}{n}\right]}$

$$= |L|^{\frac{1}{n}} e^{i\left[\frac{\theta}{n} + \frac{2k\pi}{n} + \frac{2\cancel{n}\pi}{\cancel{n}}\right]}$$

$$= |L|^{\frac{1}{n}} e^{i\left[\frac{\theta}{n} + \frac{2k\pi}{n} + 2\pi\right]}$$

But $\forall \lambda \in \mathbb{R}$, $e^{i(\lambda + 2\pi)} = e^{i\lambda}$ (Exponential is 2π-periodic, because cos and sin are 2π-periodic)

We then get: $z_{k+n} = |L|^{\frac{1}{n}} e^{i\left[\frac{\theta}{n} + \frac{2k\pi}{n}\right]} = z_k$

i.e

$$\ldots z_{n-1} = z_{-1} \quad z_0 \quad z_1 \ldots \quad z_{n-1} \quad z_n = z_0 \ldots$$

That is why $k \in \{0, 1, \ldots, n-1\}$.

Not Less than n

Now, we know that

$$z_k = |L|^{\frac{1}{n}} e^{i\left(\frac{\theta}{n} + \frac{2k\pi}{n}\right)} \quad ; \; k = 0, 1, \ldots, n-1$$

We will show that among those roots, there are no two that are equal to each other. In other words:

$$\forall k, k' \in \{0, 1, \ldots, n-1\}, \; z_k \neq z_{k'}$$

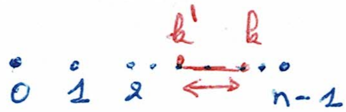
By the absurd, suppose the contrary, that, $\exists k, k' \in \{0, 1, \ldots, n-1\}$ Such that: $z_k = z_{k'}$

So $\cancel{|L|^{\frac{1}{n}}} e^{i\left(\frac{\theta}{n} + \frac{2k\pi}{n}\right)} = \cancel{|L|^{\frac{1}{n}}} e^{i\left(\frac{\theta}{n} + \frac{2k'\pi}{n}\right)}$

$$\iff \cancel{\frac{\theta}{n}} + \frac{2k\pi}{n} = \cancel{\frac{\theta}{n}} + \frac{2k'\pi}{n} + 2k''\pi \quad (k'' \in \mathbb{Z})$$

$$\iff k - k' = k'' n \quad (k'' \in \mathbb{Z})$$

One has $0 \leq k - k' \leq n$



So $\quad k - k' = k'' n \quad (k'' \in \mathbb{Z}) \Rightarrow k - k' = 0 \quad \Rightarrow k = k'$

[$k-k'$ is the distance betwen k and k', it is less or equal to n it cannot be a multiple of n, unless it is equal to zero]

$$k = k' \Rightarrow z_k = z_{k'} .$$

We just prooved the following theorem.

Theorem: Let $L \in \mathbb{C}^* / \ L = |L| e^{i\theta}$, $\theta \in \mathbb{R}$.
Let $n \in \mathbb{N}^*$.
L admits exactly n nth root of the form:

$$Z_k = |L|^{\frac{1}{n}} e^{i\left(\frac{\theta}{n} + \frac{2k\pi}{n}\right)}, \quad k \in \{0, 1, 2, \dots, n-1\}$$

Example: Find the cubic roots of $2 \cdot e^{i\frac{\pi}{7}}$.

So, we are looking for a complex $z = |z| e^{i\alpha}$

$$z^3 = 2 e^{i\frac{\pi}{7}} \Leftrightarrow |z|^3 e^{3i\alpha} = 2 e^{i\frac{\pi}{7}}$$

$$\Leftrightarrow \begin{cases} |z| = 2^{\frac{1}{3}} \\ 3\alpha = \frac{\pi}{7} + 2k\pi \ ; \ k \in \{0, 1, 2\}. \end{cases}$$

$$\Leftrightarrow \begin{cases} |z| = 2^{\frac{1}{3}} \\ \alpha = \frac{\pi}{21} + \frac{2k\pi}{3} \ ; \ k \in \{0, 1, 2\} \end{cases}$$

Solutions are:

$$\left.\begin{array}{l} k=0 \rightsquigarrow z_1 = 2^{\frac{1}{3}} e^{i\frac{\pi}{21}} \\ k=1 \rightsquigarrow z_2 = 2^{\frac{1}{3}} e^{i\left(\frac{\pi}{21} + \frac{2\pi}{3}\right)} \\ k=2 \rightsquigarrow z_3 = 2^{\frac{1}{3}} e^{i\left(\frac{\pi}{21} + \frac{4\pi}{3}\right)} \end{array}\right\} \Rightarrow S = \{z_1, z_2, z_3\}.$$

Application Find the nth roots of 1.

The solutions of the equation $Z^{n-1} + Z^{n-2} + \dots + Z + 1 = 0$

Correction

•/ We want to solve $Z^n = 1$

Put $Z = |Z| e^{i\theta}$ and $1 = (1) e^{i0}$

$$\Rightarrow \begin{cases} |z|^n = 1 \\ n\cdot\theta = 0 + 2k\pi \; ; \quad k \in \{0, 1, \dots, n-1\}. \end{cases}$$

$$\Rightarrow \begin{cases} |z| = 1 \quad (\text{because } |z| \in \mathbb{R}^+) \\ \theta = \frac{2k\pi}{n}, \quad k \in \{0, 1, 2, \dots n-1\}. \end{cases}$$

The set of the nth roots of 1 is given by:

$$S = \left\{ e^{i\frac{2k\pi}{n}}, \; k = 0, 1, \dots, n-1 \right\} = \left\{ 1, e^{i\frac{2\pi}{n}}, e^{i\frac{4\pi}{n}}, \dots \right\}$$

./ Put $P(z) = z^{n-1} + z^{n-2} + \dots + z + 1$

when $z \neq 1$; $P(z) = \frac{z^n - 1}{z - 1}$; so $P(z) = 0 \iff \frac{z^n - 1}{z - 1} = 0$

$$\Rightarrow z^n - 1 = 0 .$$

The set of solutions to the equation $P(z) = 0$ is $S - \{1\}$.

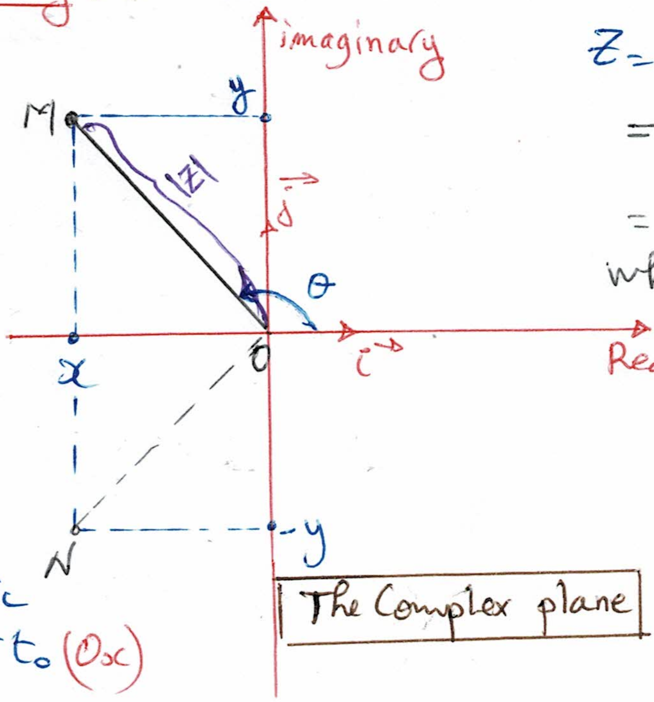
4/ Complex geometry : $\mathbb{C} = \mathbb{R}^2$

⊛ $|z|$ is the distance between O and M (always ≥ 0)

⊛ $\arg z$ is the angle $(\vec{Ox}, \vec{OM})$

⊛ M is the image of z. we note $M(z)$.

⊛ N is the image of $\bar{z}$ $N(\bar{z})$. It is the symetric of $M(z)$ with respect to (Ox)



The Complex plane

$$z = x + iy = |z|(\cos\theta + i\sin\theta) = |z| e^{i\theta}$$

where $|z| = \sqrt{x^2 + y^2}$

$$\begin{cases} \cos\theta = \frac{x}{|z|} \\ \sin\theta = \frac{y}{|z|} \\ tg\,\theta = \frac{y}{x} \quad (x \neq 0) \end{cases}$$

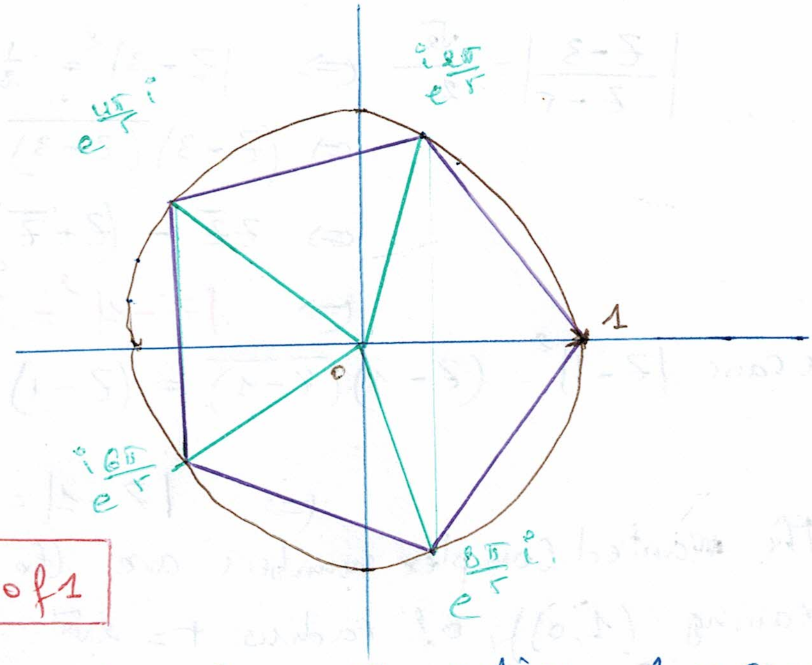
⊛ The set of points $M(z) \, / \, |z - z_A| = r \, (\geq 0)$ is the circle centered in A and radius r.

Example Represent the fifth roots of 1 on the complex plane.

we have already calculated the nth roots of 1

So the fifth roots are given by:

$$\left\{ 1, e^{i\frac{2\pi}{5}}, e^{i\frac{4\pi}{5}}, e^{i\frac{6\pi}{5}}, e^{i\frac{8\pi}{5}} \right\}$$



The 5^{th} roots of 1

Remarks: ① The n^{th} roots of 1 form the vertices of a regular Polygon of n sides, inscribed in the unit circle of the Complex plane, with a vertex at trivial root 1.

② When $n \geq 2$, The sum of the nth roots of 1 is equal to 0.
For instance, Let's Calculate the sum of the fifth roots of 1,

$$S = 1 + e^{i\left(\frac{2\pi}{5}\right)} + e^{i\frac{6\pi}{5}} + e^{i\frac{6\pi}{5}} + e^{i\frac{8\pi}{5}} = 1 + e^{i\frac{2\pi}{5}} + \left(e^{i\frac{2\pi}{5}}\right)^2 + \left(e^{i\frac{2\pi}{5}}\right)^3 + \left(e^{i\frac{2\pi}{5}}\right)^4$$

we know that $\sum_{k=0}^{n} z^k = 1 + z + z^2 + \cdots + z^n = \frac{1 - z^n}{1 - z}$, So

$$S = \frac{1 - \left(e^{i\frac{2\pi}{5}}\right)^5}{1 - e^{i\frac{2\pi}{5}}} = \frac{1 - \overbrace{e^{i2\pi}}^{=1}}{1 - e^{i\frac{2\pi}{5}}} = 0 .$$

Exercises at home and the amphi !

1/ Find all the complex numbers such that: $\left|\frac{z-3}{z-5}\right| = \frac{\sqrt{2}}{2}$.

2/ knowing that: $\forall z, w \in \mathbb{C}^*$

$$Arg(z) = Arg(w) \ [2\pi] \iff \exists \lambda \in \mathbb{R}_+^* \ / \ z = \lambda w$$

Find all Complex numbers z such that:

$$\begin{cases} |z| = |z - 2| \\ Arg(z) = Arg(z + 3 + i) \end{cases}$$